On Certain Admissible Embeddings of L-groups

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Abstract

Let F be a local field and E/F be a separable extension of degree n. Regard $T = \mathrm{Res}_{E/F}\mathbb{G}_m$ as an elliptic maximal torus of $G = \mathrm{GL}_n$. We can construct an admissible embedding of L-groups ${}^LT \hookrightarrow {}^LG$ using Langlands-Shelstad χ-data. Such embedding gives rise to an induced representation of the Weil group W_F of F from a character of W_E. The relation between induced representations and admissible embeddings provides a different interpretation of the work of Bushnell-Henniart on the essentially tame local Langlands correspondence.

1 Introduction

Let F be a local field and W_F be the Weil group of F. Let G be GL_n as a reductive algebraic group over F. Let E be a field extension of degree n. We may regard $G(F)$ as the automorphism group of the F-vector space E by choosing an F-basis of E. Write $E^\times$ as the F-point of the algebraic torus $T = \mathrm{Res}_{E/F}\mathbb{G}_m$. Therefore the multiplicative action of T on E gives rise to an F-embedding $T \hookrightarrow G$.

We consider the dual problem as follows. Let $\hat{G}$ and $\hat{T}$ be the dual groups of G and T. Let ${}^LG$ and ${}^LT$ be the corresponding L-groups. By fixing a maximal torus $\mathcal{T}$ in $\hat{G}$, we ask whether there exists an admissible embedding ${}^LT \hookrightarrow {}^LG$, an injective morphism of groups that maps $\hat{T}$ bijectively onto $\mathcal{T}$ and the W_F-component of ${}^LT$ identically to that of ${}^LG$. The answer is affirmative and a construction is given by [LS87].

We introduce the idea briefly as follows. The problem can be shown to be equivalent to ask whether the exact sequence $1 \to \mathcal{T} \to {}^LT \to W_F \to 1$ splits, and is therefore equivalent to ask wether the cohomology class $t = t({}^LT) \in H^2(W_F, \mathcal{T})$ defined by ${}^LT$ is trivial or not. We can construct a splitting for the class t using a collection of characters $\{\chi_\lambda\}_\lambda$ called χ-data. Here λ runs through $\mathcal{R}(G,T)$ the root system of the maximal torus T in G, and the character χ_λ is defined on the multiplicative group of a field extension E_λ over some Galois conjugate of E. We shall go over the properties of χ-data and construct the corresponding admissible embedding in Section 3.

The first main result of this article is a relation between admissible embedding and induced representation of Weil group. The 1-cohomology group $H^1(W_F, \hat{T}) = \mathrm{Int}(\hat{T})\backslash \mathrm{Hom}_{W_F}(W_F, \hat{T} \rtimes W_F)$ is isomorphic to $\mathrm{Hom}(W_E, \mathbb{C}^\times)$ naturally. (This fact is known as Shapiro's Lemma.) Using this fact and Proposition 2.3, the set $\mathrm{Int}(\mathcal{T})\backslash \mathrm{AE}({}^LT, {}^LG)$ of $\mathrm{Int}(\mathcal{T})$-equivalence classes of admissible embeddings ${}^LT \to {}^LG$ is a $\mathrm{Hom}(W_E, \mathbb{C}^\times)$-torsor. By specifying suitable isomorphism between these bijective sets, we have the following.

Theorem 1.1 (Proposition 2.5)**.** *Suppose $\tilde{\xi} \in \mathrm{Hom}_{W_F}(W_F, \hat{T} \rtimes W_F)$ and $\chi \in \mathrm{AE}({}^LT, {}^LG)$ correspond to characters ξ and μ in $\mathrm{Hom}(W_E, \mathbb{C}^\times)$ respectively. The composition $\chi \circ \tilde{\xi}$ when projected to $\hat{G} = \mathrm{GL}_n(\mathbb{C})$ is isomorphic to $\mathrm{Ind}_{W_E}^{W_F}(\xi\mu)$ as representations of W_F.*

The second main result is a converse of the first one. Suppose now $\chi \in \mathrm{AE}({}^LT, {}^LG)$ is defined by a collection of χ-data $\{\chi_\lambda\}$. We can recover the character μ, in terms of $\{\chi_\lambda\}$, that induces the representation $\mathrm{Ind}_{W_E}^{W_F}(\xi\mu) \cong \chi \circ \tilde{\xi}$.

Theorem 1.2 (Proposition 4.2)**.** *Suppose the admissible embedding $\chi : {}^LT \to {}^LG$ is defined by χ-data $\{\chi_\lambda\}$. The character μ in Theorem 1.1 can be taken to be*

$$\mu = \prod_{\text{certain } \lambda} \mathrm{Res}_{E^\times}^{E_\lambda^\times} \chi_\lambda.$$

The proof comes from a comparison between the expression of χ given by the recipe in [LS87] and the matrix coefficients of $\mathrm{Ind}_{W_E}^{W_F}(\xi\mu)$. What missing in Theorem 1.2 is the set through which λ runs in the product form of μ. As suggested by the expression of each factor, we take those roots λ whose corresponding field extensions E_λ contain E. These roots form a set of representatives of the W_F-orbits of the root system $\mathcal{R}(G,T)$. We will be more specific on these representatives using a double coset expression in Proposition 4.1. We emphasis that such μ cannot be arbitrary. For example, μ satisfies $\mu|_{F^\times} = \det \mathrm{Ind}_{W_E}^{W_F} 1_{W_E}$, as shown in Proposition 4.7.

Finally we give an application on a particular case of the local Langlands correspondence, known as the essentially tame case, established in [BH05, BH10]. Let F be a non-Archimedean local field of characteristic 0, and E be a tamely ramified extensions over F of degree n. For each admissible character [BH05] ξ of $E^\times$, we introduce a character ${}_F\mu_\xi$ of $E^\times$, called the rectifier of ξ. Its purpose is to measure the difference between a 'naive' version of the local Langlands Correspondence [BH05] and the essentially tame one. In a subsequent article [Tam], we prove that the rectifier admits a factorization of the form in Theorem 1.2 with canonical choices of the characters $\{\chi_\lambda\}$. In other words, we can express the essentially tame local Langlands Correspondence by admissible embeddings constructed by χ-data.

Outline of the Article In Section 2 we study the induced representation of Weil group by admissible embedding of L-groups and prove Theorem 1.1. To construct an admissible embedding in general we need χ-data, whose definition and properties are discussed in Section 3. We prove the main result Theorem 1.2 and some related facts in Section 4. Finally in Section 5 we describe providently how admissible embedding is related to the essentially tame local Langlands Correspondence.

Notations We fix our notations throughout the article. Let H be a group and K be a subgroup of H. The normalizer of K in H is denoted by $N_H(K)$. Suppose H acts on a set X. For $h \in H$ and $x \in X$, we write ${}^{h_X}x$, or simply ${}^h x$, for the action of h on x. The H-orbit of $x \in X$ is denoted by ${}^H x$. The collection of all H-orbits of X is denoted by $H\backslash X$. The set of fixed points is denoted by X^H. If f is a map whose domain is X, we write ${}^h f(x) = f^{h^{-1}}(x) = f({}^{h^{-1}}x)$. If X is an abelian group, we denote the set of j-cocycle of H with values in X by $Z^j(H,X)$, and the j-cohomology group by $H^j(H,X)$.

Given a field extension E/F and the corresponding Weil groups $W_E \subseteq W_F$, we denote induction $\mathrm{Ind}_{W_E}^{W_F}$ by $\mathrm{Ind}_{E/F}$ and restriction $\mathrm{Res}_{W_E}^{W_F}$ by $\mathrm{Res}_{E/F}$. For $G = \mathrm{GL}_n$ as an F-group we define

$$\hat{G} = \mathrm{GL}_n(\mathbb{C}) \quad \text{and} \quad {}^L G = \mathrm{GL}_n(\mathbb{C}) \times W_F,$$

namely the dual-group and the L-group of G. Given a field extension E/F, let T be the F-torus $T = \mathrm{Res}_{E/F}\mathbb{G}_m$ with

$$\hat{T} = \mathrm{Ind}_{E/F}\mathbb{C}^\times = (\mathbb{C}^\times)^{[E/F]} \quad \text{and} \quad {}^L T = \hat{T} \rtimes W_F$$

as its dual-group and L-group. Denote the root system of T in G by $\mathcal{R}(G,T)$ and the corresponding Weyl group by $\Omega(G,T)$.

2 Induction and Admissible Embedding

Let G be a connected reductive algebraic group defined and quasi-split over F. Let T be a maximal torus of G also defined over F.

Definition 2.1. An *admissible embedding* from ${}^L T$ to ${}^L G$ is a morphism of groups $\chi : {}^L T \to {}^L G$ of the form

$$\chi(t \rtimes w) = \iota(t)\bar{\chi}(w) \rtimes w$$

for some injective morphism $\iota : \hat{T} \to \hat{G}$ and some map $\bar{\chi} : W_F \to \hat{G}$. □

By expanding $\chi(s \rtimes v)\chi(t \rtimes w) = \chi((s \rtimes v)(t \rtimes w))$, we can show that

$$(\mathrm{Int}\bar{\chi}(v))({}^{v_{\hat{G}}}\iota(t)) = \iota({}^{v_{\hat{T}}}t) \text{ and } \bar{\chi}(vw) = \bar{\chi}(v){}^{v_{\hat{G}}}\bar{\chi}(w) \tag{1}$$

for all $t \in \hat{T}$ and $v, w \in W_F$. Hence $\bar{\chi}$ has image in $N_{\hat{G}}(\iota(\hat{T}))$. Conversely if ι and $\bar{\chi}$ satisfy (1), then the map in Definition 2.1 is an admissible embedding. We can rephrase (1) as follows. Let $N_{\hat{G}}(\iota(\hat{T})) \rtimes W_F$ acts on $\hat{T}$ by ${}^{x\rtimes w}t = \iota^{-1}(\mathrm{Int}(x)({}^{w_{\hat{G}}}\iota(t)))$, then the morphism $W_F \to \mathrm{Aut}(\hat{T})$, $w \mapsto w_{\hat{T}}$ factors through

$$W_F \to N_{\hat{G}}(\iota(\hat{T})) \rtimes W_F,\ w \mapsto \bar{\chi}(w) \rtimes w.$$

Let $\mathcal{H}$ be a subgroup of $\hat{G}$. Two admissible embeddings χ_1, χ_2 are called $\mathrm{Int}(\mathcal{H})$-equivalent if there is $x \in \mathcal{H}$ such that $\chi_1(t \rtimes w) = (x \rtimes 1)\chi_2(t \rtimes w)(x \rtimes 1)^{-1}$ for all $t \rtimes w \in {}^LT$. Using (1) we can show that this condition is equivalent to require an $x \in \mathcal{H}$ giving $\bar{\chi_1}(w) = x\bar{\chi_2}(w)^{w_{\hat{G}}}x^{-1}$ for all $w \in W_F$.

Let's provide more preliminary setup. By taking a conjugate of T in G, which is still denoted by T for brevity, let T be contained in a Borel subgroup B defined over F. Choose an W_F-invariant splitting $(\mathcal{T}, \mathcal{B}, \{\hat{X}_\alpha\})$ of $\hat{G}$ and an isomorphism $(\hat{T}, \hat{B}) \to (\mathcal{T}, \mathcal{B})$ whose restriction on $\hat{T}$ is ι. For notation convenience we usually omit ι and write $t = \iota(t) \in \mathcal{T}$ for $t \in \hat{T}$, but bear in mind that $\hat{T}$ and $\mathcal{T}$ may have inequivalent W_F-actions.

Remark 2.2. We choose splittings on G and $\hat{G}$ so that we have a duality on the bases of T and $\mathcal{T}$ for explicit computations. For example, we choose a basis of $\mathcal{T}$ for the construction of the Steinberg section (see Section 2.1 of [LS87]). Our main results would be independent of these choices. For instance, the $\hat{G}$-conjugacy class of an admissible embedding is independent of the choices of the Borel subgroup B containing T and the splitting $(\mathcal{T}, \mathcal{B}, \{\hat{X}_\alpha\})$ of $\hat{G}$ (see [LS87] (2.6.1) and (2.6.2)). □

Langlands and Shelstad constructed a particular form of admissible embedding using χ-data (see [LS87] (2.5)). We will give the construction in Section 3. Let's assume such construction for a moment, and denote $\mathrm{AE}({}^LT, {}^LG, (\hat{T}, \hat{B}) \to (\mathcal{T}, \mathcal{B}))$, or just $\mathrm{AE}({}^LT, {}^LG)$ for simplicity, the set of admissible embeddings ${}^LT \to {}^LG$ (associated to the choice of the isomorphism $(\hat{T}, \hat{B}) \to (\mathcal{T}, \mathcal{B})$). The description of this collection is not difficult.

Proposition 2.3. *The set* $\mathrm{AE}({}^LT, {}^LG)$ *is a* $Z^1(W_F, \hat{T})$*-torsor, and the set of its* $\mathrm{Int}(\mathcal{T})$*-equivalence classes* $\mathrm{Int}(\mathcal{T})\backslash\mathrm{AE}({}^LT, {}^LG)$ *is an* $H^1(W_F, \hat{T})$*-torsor.*

Proof. We fix an embedding $\chi_0 \in \mathrm{AE}({}^LT, {}^LG)$ and take $\bar{\chi}_0 : W_F \to \hat{G}$ as in Definition 2.1. Then for each $\chi \in \mathrm{AE}({}^LT, {}^LG)$, the difference $\bar{\chi}\bar{\chi_0}^{-1}$ is a 1-cocycle of W_F valued in $\hat{T}$, i.e. $\bar{\chi}\bar{\chi_0}^{-1} \in Z^1(W_F, \hat{T})$. Indeed for a fixed $w \in W_F$ both $\bar{\chi}(w)$ and $\bar{\chi_0}(w)$ project to the same element in $\Omega(\hat{G}, \mathcal{T}) = N_{\hat{G}}(\mathcal{T})/\mathcal{T}$. Using (1) we have that

$$\begin{aligned} \bar{\chi}\bar{\chi_0}^{-1}(vw) &= \bar{\chi}(v)^{v_{\hat{G}}}\bar{\chi}(w)^{v_{\hat{G}}}\bar{\chi_0}(w)^{-1}\bar{\chi_0}(v)^{-1} \\ &= \bar{\chi}(v)\bar{\chi_0}(v)^{-1\,v_{\hat{T}}}\bar{\chi}(w)^{v_{\hat{T}}}\bar{\chi_0}(w)^{-1} \end{aligned}$$

for all $v, w \in W_F$. We can readily verify that the map

$$\mathrm{AE}({}^LT, {}^LG) \to Z^1(W_F, \hat{T}),\ \chi \mapsto \bar{\chi}\bar{\chi_0}^{-1}$$

is bijective. From the equality $t\bar{\chi}(v)^{v_{\hat{G}}}t^{-1}\bar{\chi_0}^{-1}(v) = t\bar{\chi}(v)\bar{\chi_0}^{-1}(v)^{v_{\hat{T}}}t^{-1}$ for all $t \in \mathcal{T}$, we know that two embeddings are $\mathrm{Int}(\mathcal{T})$-equivalent if and only if the corresponding 1-cocycles differ by a coboundary in $Z^1(W_F, \hat{T})$. □

Remark 2.4. For $G = \mathrm{GL}_n$ we can construct an explicit embedding ${}^LT \to {}^LG$. Choose $\mathcal{T}$ to be the diagonal subgroup of $\hat{G}$. We embed $\hat{T}$ into $\hat{G}$ with image $\mathcal{T}$ and define

$$W_F \to N_{\hat{G}}(\hat{T}),\ w \mapsto N(w)$$

the permutation matrix whose assignment is according to the W_F-action on $\hat{T} \cong \mathbb{C}^{[E/F]}$, i.e. $\mathrm{Int}(N(v))t = {}^{v_{\hat{T}}}t$ for all $t \in \hat{T}$. Clearly the map ${}^LT \to {}^LG$, $t \rtimes w \mapsto tN(w) \times w$ defines an admissible embedding. □

For $G = \mathrm{GL}_n$ and $T = \mathrm{Res}_{E/F}\mathbb{G}_m$ we give the main result Proposition 2.5 of this section after the following setup. By Shapiro's Lemma (see the Exercise in [Ser79] VII §5.), we have a special case of Langlands Correspondence for torus

$$\mathrm{Hom}(E^\times, \mathbb{C}^\times) = H^1(W_F, \hat{T}). \tag{2}$$

The precise correspondence is given as follows. Suppose ξ is a character of $E^\times$. We regard ξ as a character of W_E by class field theory [Tat79]. Take a collection of coset representatives $\{g_1, \dots, g_n\}$ of $W_E \backslash W_F$. Define for each g_i a map $u_{g_i} : W_F \to W_E$ given by

$$g_i w = u_{g_i}(w) g(g_i, w) \quad \text{for } g(g_i, w) \in \{g_1, \dots, g_n\}. \tag{3}$$

Then define

$$\tilde{\xi} : W_F \to \hat{T} \cong \mathbb{C}^n,\, w \mapsto (\xi(u_{g_1}(w)), \dots, \xi(u_{g_n}(w)))\,.$$

It can be checked that $\tilde{\xi}$ is a 1-cocycle in $Z^1(W_F, \hat{T})$, and different choices of coset representatives give cocycles different from $\tilde{\xi}$ by a 1-coboundary. Hence the 1-cohomology class of $\tilde{\xi}$ is defined. By abusing of language, we call $\tilde{\xi}$ a *Langlands parameter* of ξ. Moreover, combining Proposition 2.3 and (2) we have

$$\operatorname{Hom}(E^\times, \mathbb{C}^\times) = \operatorname{Int}(\mathcal{T}) \backslash \operatorname{AE}({}^L T, {}^L G). \tag{4}$$

Explicitly, if we have a character μ of $E^\times$, then define

$$\chi : {}^L T \to {}^L G,\, t \rtimes w \mapsto t \begin{pmatrix} \mu(u_{g_1}(w)) & & \\ & \ddots & \\ & & \mu(u_{g_n}(w)) \end{pmatrix} N(w) \times w.$$

Here $N(w)$ is the permutation matrix as introduced in Remark 2.4. Notice that this bijection is non-canonical. Write $\operatorname{proj} : {}^L G \to \hat{G}$, $g \times w \mapsto g$, which is a morphism of groups because $G = \mathrm{GL}_n$ splits over F. Combining the bijections (2) and (4), we have the following result.

Proposition 2.5. *Suppose that ξ and μ come from $\tilde{\xi}$ and χ by the bijections (2) and (4). The composition*

$$H^1(W_F, \hat{T}) \times \operatorname{Int}(\mathcal{T}) \backslash \operatorname{AE}({}^L T, {}^L G) \to \operatorname{Int}(\hat{G}) \backslash \operatorname{Hom}_{W_F}(W_F, {}^L G)$$

such that $(\tilde{\xi}, \chi) \mapsto \chi \circ \tilde{\xi}$ gives an isomorphism $\operatorname{proj} \circ \chi \circ \tilde{\xi} \cong \operatorname{Ind}_{E/F}(\xi\mu)$ as representations of W_F.

Proof. Choose a suitable basis on the representation space of $\operatorname{Ind}_{E/F}(\xi\mu)$. For example, if we realize our induced representation by the subspace of functions

$$\{f : W_F \to \mathbb{C} | f(xg) = \xi\mu(x) f(g) \text{ for all } x \in W_E, g \in W_F\},$$

then we choose those f_i determined by $f_i(g_j) = \delta_{ij}$ (Kronecker delta) as basis vectors. The matrix coefficient of $\operatorname{Ind}_{E/F}(\xi\mu)$ is therefore

$$\begin{pmatrix} \mu\xi(u_{g_1}(w)) & & \\ & \ddots & \\ & & \mu\xi(u_{g_n}(w)) \end{pmatrix} N(w)$$

the same matrix as the image of $\chi \circ \tilde{\xi}$. $\square$

Remark 2.6. We can recover ξ from $\operatorname{Ind}_{E/F}\xi$ as follows. We choose the first k coset representatives $g_1 = 1, g_2 \dots, g_k$ to be those in the normalizer $N_{W_F}(W_E) = \operatorname{Aut}_F(E)$, and (by choosing suitable basis) consider the matrix coefficient of $\operatorname{Res}_{E/F}\operatorname{Ind}_{E/F}\xi$. The first k diagonal entries are always non-zero and give the characters ξ^{g_i}. $\square$

3 Langlands-Shelstad χ-data

In this section we recall the construction of admissible embeddings ${}^L T \to {}^L G$ given in Chapter 2 of [LS87]. Here G is a connected reductive algebraic group defined and quasi-split over F and T is a maximal torus in G also defined over F. Take a maximal torus $\mathcal{T}$ of $\hat{G}$ and choose a splitting $(\mathcal{T}, \mathcal{B}, \{\hat{X}_\alpha\})$ for $\hat{G}$. We again emphasis that different choices yield $\operatorname{Int}(\hat{G})$-equivalent embeddings. For computational convenience we choose $\mathcal{T}$ to be the diagonal group and $\mathcal{B}$ to be the group of upper triangular matrices. The tori $\hat{T}$ and $\mathcal{T}$ are isomorphic as groups but with different W_F-actions.

Recall that the existence of an admissible embedding $\chi : {}^LT \to {}^LG$ with restriction $\hat{T} \to \mathcal{T}$ is equivalent to the existence of a 1-cocycle $\bar{\chi} \in Z^1(W_F, N_{\hat{G}}(\mathcal{T}))$ as in (1). We write

$$\omega : W_F \to \Omega(\hat{G}, \mathcal{T}) \rtimes W_F,\ w \mapsto \omega(w) = \bar{\omega}(w) \rtimes w$$

such that the action of $\omega(w)$ on $t \in \hat{T}$ is the same as ${}^{w_{\hat{T}}}t$. We recall in (2.1) of [LS87] the Steinberg section $n_{\mathrm{St}} : \Omega(\hat{G}, \mathcal{T}) \to N_{\hat{G}}(\mathcal{T})$ and define

$$n : W_F \to N_{\hat{G}}(\mathcal{T}) \rtimes W_F,\ w \mapsto n(w) = \bar{n}(w) \rtimes w := n_{\mathrm{St}}(\bar{\omega}(w)) \rtimes w.$$

The map n may not be a morphism of groups, yet $\bar{n}$ satisfies the first equation in (1) in place of $\bar{\chi}$. We write

$$t_{\mathrm{b}}(v, w) = n(v)n(w)n(vw)^{-1} = \bar{n}(v)^{v_{\hat{G}}}\bar{n}(w)\bar{n}(vw)^{-1}, \tag{5}$$

a 2-cocycle of W_F, whose values are in $\{\pm 1\}^n \subseteq \mathcal{T}$ by Lemma 2.1.A of [LS87]. Hence the problem of seeking such $\bar{\chi}$ is equivalent to looking for a map $r_{\mathrm{b}} : W_F \to \hat{T}$ that splits t_{b}^{-1}, i.e.

$$r_{\mathrm{b}}(v)^{v_{\hat{T}}} r_{\mathrm{b}}(w) r_{\mathrm{b}}(vw)^{-1} = t_{\mathrm{b}}(v, w)^{-1}. \tag{6}$$

Remark 3.1. If we regard a group $\mathcal{H}$ which appears in the exact sequence $1 \to \mathcal{T} \to \mathcal{H} \to W_F \to 1$ as a subgroup of $N_{\hat{G}}(\mathcal{T}) \rtimes W_F$, then the problem whether such splitting r_{b} exists is equivalent to ask whether the exact sequence above with $\mathcal{H} \cong {}^LT$ splits or not. □

The idea in [LS87] to construct such splitting r_{b} is to choose a set of characters called χ-data, which is defined after the following setup. For each λ in the root system $\mathcal{R} = \mathcal{R}(G, T)$, we denote the stabilizers

$$W_{+\lambda} = \{w \in W_F | w\lambda = \lambda\} \quad \text{and} \quad W_{\pm\lambda} = \{w \in W_F | w\lambda = \pm\lambda\},$$

and fixed fields

$$E_{+\lambda} = \bar{F}^{W_{+\lambda}} \quad \text{and} \quad E_{\pm\lambda} = \bar{F}^{W_{\pm\lambda}}.$$

In general, $E_{+\lambda}$ is a field extension of some conjugate of E. We call a root λ *symmetric* if $|E_{+\lambda}/E_{\pm\lambda}| = 2$, and *asymmetric* otherwise. By definition this symmetry is preserved by the W_F-action. Let

(i) $W_F \backslash \mathcal{R}_{sym}$ be the set of W_F-orbits of symmetric roots,

(ii) $W_F \backslash \mathcal{R}_{asym}$ be the set of W_F-orbits of asymmetric roots, and

(iii) $W_F \backslash \mathcal{R}_{asym/\pm}$ be the set of equivalent classes of asymmetric W_F-orbits by identifying ${}^{W_F}\lambda$ and ${}^{W_F}(-\lambda)$.

Definition 3.2. We define a collection of characters $\{\chi_\lambda : E_{+\lambda}^\times \to \mathbb{C}^\times | \lambda \in \mathcal{R}\}$, called χ-*data*, such that the following conditions hold.

(i) For each $\lambda \in \mathcal{R}$, we have $\chi_{-\lambda} = \chi_\lambda^{-1}$ and $\chi_{w\lambda} = \chi_\lambda^{w^{-1}}$ for all $w \in W_F$.

(ii) If λ is symmetric, then $\chi|_{E_{\pm\lambda}^\times}$ equals the quadratic character $\delta_{E_{+\lambda}/E_{\pm\lambda}}$ attached to the extension $E_{+\lambda}/E_{\pm\lambda}$.

□

If we choose $\mathcal{R}_0$ to be a subset of $\mathcal{R}$ consisting of representatives of $W_F \backslash \mathcal{R}_{sym}$ and $W_F \backslash \mathcal{R}_{asym/\pm}$, then by condition (i) it is enough to define a collection of χ-data on $\mathcal{R}_0$.For a chosen χ-data $\{\chi_\lambda\}_{\lambda \in \mathcal{R}_0}$, following (2.5) of [LS87] we define for each $\lambda \in \mathcal{R}_0$ a map

$$r_\lambda : W_F \to \mathcal{T}, \quad w \mapsto \prod_{g_i \in W_{\pm\lambda} \backslash W_F} \chi_\lambda(v_1(u_{g_i}(w)))^{g_i^{-1}\lambda}, \tag{7}$$

where u_{g_i} is the map (3) for $W_{\pm\lambda} \backslash W_F$ and v_1 is defined similarly for $W_{+\lambda} \backslash W_{\pm\lambda}$. We then define

$$r_{\mathrm{g}} = \prod_{\lambda \in \mathcal{R}_0} r_\lambda. \tag{8}$$

Such construction yields (Lemma 2.5.A of [LS87]) a 2-cocycle

$$t_{\mathrm{g}}(v,w) = r_{\mathrm{g}}(v)^{v_{\hat{T}}} r_{\mathrm{g}}(w) r_{\mathrm{g}}(vw)^{-1} \in Z^2(W_F, \{\pm 1\}^n). \tag{9}$$

In constructing the 2-cocycles (5) and (9) we implicitly used two different notions of gauges (defined just before Lemma 2.1.B of [LS87]) on the set $\mathcal{R}$. To relate them we introduce a map (see (2.4) of [LS87]) $s = s_{\mathrm{b/g}} : W_F \to \{\pm 1\}^n$ such that

$$s(v)^{v_{\hat{T}}} s(w) s(vw)^{-1} = t_{\mathrm{b}}(v,w) t_{\mathrm{g}}(v,w)^{-1}. \tag{10}$$

Write $r_{\mathrm{b}} = s_{\mathrm{b/g}} r_{\mathrm{g}}$ and $\bar{\chi} = r_{\mathrm{b}} \bar{n}$.

Proposition 3.3. *The map χ defines an admissible embedding ${}^LT \to {}^LG$.*

Proof. It suffices to show that $\bar{\chi}$ satisfies the two conditions in (1). The first condition is just from the definition of $n(w)$, while the second condition is a straightforward calculation using (5), (6), (9), and (10). $\square$

4 The Main Results

Let $G = \mathrm{GL}_n$ and $T = \mathrm{Res}_{E/F}\mathbb{G}_m$, both regarded as algebraic groups over F. Any root in the root system $\mathcal{R} = \mathcal{R}(G,T)$ can be expressed as $\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]$ for some $g, h \in W_F$, with

$$\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](t) = {}^{g^{-1}}t({}^{h^{-1}}t)^{-1}$$

for all $t \in E^\times$. The W_F-action on $\mathcal{R}$ is given by ${}^w\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right] = \left[\begin{smallmatrix} gw^{-1} \\ hw^{-1} \end{smallmatrix}\right]$ for $w \in W_F$. Notice that such action factors through the one of the Weyl group $\Omega(G,T)$. If we choose a collection of coset representatives $\{g_1 = 1, g_2, \dots, g_n\}$ of $W_E\backslash W_F$, then we can write $\lambda = \left[\begin{smallmatrix} g_i \\ g_j \end{smallmatrix}\right]$ with $i \neq j$. It is clear that each orbit ${}^{W_F}\lambda$ contains a root of the form $\left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right]$ for some $g \in \{g_2, \dots, g_n\}$.

Proposition 4.1. *The set $W_F\backslash\mathcal{R}$ of W_F-orbits of the root system $\mathcal{R}$ is bijective to the collection of non-trivial double cosets in $W_E\backslash W_F/W_E$, by*

$$W_F\backslash\mathcal{R} \to (W_E\backslash W_F/W_E) - \{W_E\},\ {}^{W_F}\lambda = {}^{W_F}\left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right] \mapsto W_E g W_E.$$

Proof. The set of roots $\mathcal{R}$ can be identified with the set of off-diagonal elements of $W_E\backslash W_F \times W_E\backslash W_F$, with W_F-action by ${}^g(W_E g_1, W_E g_2) = (W_E g_1 g^{-1}, W_E g_2 g^{-1})$. By elementary group theory, we know that the orbits are bijective to the non-trivial double cosets in $W_E\backslash W_F/W_E$. $\square$

We denote $(W_E\backslash W_F/W_E)'$ the collection of non-trivial double cosets, and $[g]$ the double coset $W_E g W_E$. We call $g \in W_F$ *symmetric* if $[g] = [g^{-1}]$ and *asymmetric* otherwise. Clearly such symmetry descends to an analogous property on $(W_E\backslash W_F/W_E)'$. By Proposition 4.1 the symmetry of $(W_E\backslash W_F/W_E)'$ is equivalent to the symmetry of $W_F\backslash\mathcal{R}$. Let

(i) $(W_E\backslash W_F/W_E)_{sym}$ be the set of symmetric non-trivial double cosets,

(ii) $(W_E\backslash W_F/W_E)_{asym}$ be the set of asymmetric non-trivial double cosets, and

(iii) $(W_E\backslash W_F/W_E)_{asym/\pm}$ be the set of equivalent classes of $(W_E\backslash W_F/W_E)_{asym}$ by identifying $[g]$ with $[g^{-1}]$.

Let $\mathcal{D}$ to be a set of representatives in $g \in W_E\backslash W_F$ of $(W_E\backslash W_F/W_E)_{sym}$ and $(W_E\backslash W_F/W_E)_{asym/\pm}$. If $\lambda = \left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right]$, we write χ_λ as χ_g and $E_{+\lambda}$ as E_g which equals $E_g = {}^{g^{-1}}EE$. By condition (i) of Definition 3.2, a collection of χ-data $\{\chi_\lambda\}$ depends only on its sub-collection $\{\chi_g\}$ for $g \in \mathcal{D}$. We also call such sub-collection χ-data.

Given χ-data $\{\chi_g\}$ let χ be the admissible embedding defined by $\{\chi_g\}$ as in Proposition 3.3. Let ξ be a character of $E^\times$ and $\tilde{\xi} \in Z^1(W_F, \hat{T})$ be a Langlands parameter of ξ. (The choice of $\tilde{\xi}$ is known to be irrelevant.) In Proposition 2.5 we described an induced representation $\mathrm{Ind}_{E/F}\xi$ as certain embedding of the image of $\tilde{\xi}$ into $\mathrm{GL}_n(\mathbb{C})$. Here we have the reverse.

Proposition 4.2. *Given χ-data $\{\chi_g\}$, define*

$$\mu = \mu_{\{\chi_g\}} = \prod_{[g]\in(W_E\backslash W_F/W_E)'} \mathrm{Res}^{E_g^\times}_{E^\times}\chi_g.$$

Let χ be the admissible embedding defined by $\{\chi_g\}$. Then for all character ξ of $E^\times$, the composition

$$W_F \xrightarrow{\tilde{\xi}} \hat{T}\rtimes W_F \xrightarrow{\chi} \hat{G}\times W_F \xrightarrow{\mathrm{proj}} \mathrm{GL}_n(\mathbb{C})$$

is isomorphic to $\mathrm{Ind}_{E/F}(\xi\mu)$ as representations of W_F.

Remark 4.3. Notice that the product in Proposition 4.2 is uniquely determined by $\{\chi_\lambda\}$, i.e. independent of the representative $g\in\mathcal{D}$, which is itself a coset representative of $W_E\backslash W_F$, of the double coset $[g]$. Indeed if ${}^x\left[\begin{smallmatrix}1\\g\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\h\end{smallmatrix}\right]$ for some $x\in W_F$, then ${}^xE_g = E_h$ and so $\mathrm{Res}^{E_h^\times}_{E^\times}\chi_h = \mathrm{Res}^{{}^xE_g^\times}_{E^\times}\chi_g^{x^{-1}} = \mathrm{Res}^{E_g^\times}_{E^\times}\chi_g$ by condition (i) of Definition 3.2. □

Remark 4.4. Suppose we have fixed a character ξ of $E^\times$. Take a subset $\{g_1 = 1, g_2\dots, g_k\}$ of coset representatives of $W_E\backslash W_F$ in the normalizer $N_{W_F}(W_E) = \mathrm{Aut}_F(E)$ and write $\mu_1 = \mu_{\{\chi_g\}}$ as in Proposition 4.2. Then all other characters μ_k such that $\mathrm{Ind}_{E/F}(\xi\mu_k)\cong\mathrm{Ind}_{E/F}(\xi\mu_1)$ are of the form $\mu_k = \xi^{g_k-1}\mu_1$. This character μ_k also has a factorization in Proposition 4.2 with the same χ-data of μ, except when $g = g_k$ the character χ_g is changed according to the following.

(i) If g is symmetric, then χ_g is replaced by $\xi^{g-1}\chi_g$.

(ii) If g is asymmetric, then χ_g is replaced by $\xi^g\chi_g$ and so $\chi_{g^{-1}}$ by $\xi^{-1}\chi_{g^{-1}}$.

□

Proof. (of Proposition 4.2) We first abbreviate $H = W_F$ and $K = W_E$. For each $\lambda = \left[\begin{smallmatrix}1\\g\end{smallmatrix}\right]$ we denote $K_g = K\cap g^{-1}Kg$, which equals $W_{+\lambda}$. If $[g]\in(K\backslash H/K)_\pm$, then because $KgK = Kg^{-1}K$ we can replace g by an element in Kg such that $g^2\in K$. Subsequently we have $g\in W_{\pm\lambda}$ and $g^2\in K_g = W_{+\lambda}$. We denote $K_{\pm g}$ the group generated by $K\cap g^{-1}Kg$ and g, which equals $W_{\pm\lambda}$. By condition (i) of Definition 3.2, we rewrite the product in Proposition 4.2 as

$$\prod_{[g]\in(K\backslash H/K)_{asym/\pm}} (\mathrm{Res}^{E_g^\times}_{E^\times}\chi_g)(\mathrm{Res}^{{}^gE_g^\times}_{E^\times}\chi_g^{g^{-1}})^{-1} \prod_{[g]\in(K\backslash H/K)_{sym}} (\mathrm{Res}^{E_g^\times}_{E^\times}\chi_g). \tag{11}$$

Recall that our dual group $\mathcal{T}$ is the diagonal subgroup. In order to check χ gives rise to a character μ as (11) it is enough to consider the first entry of r_{g} (see (8) and the discussion in Remark 2.6). From (7) we have

$$r_{\mathrm{g}}(w) = \left(\prod_{[g]\in(K\backslash H/K)_{asym/\pm}}\prod_{g_i\in K_g\backslash H}\chi_g(u_{g_i}(w))^{\left[\begin{smallmatrix}g_i\\gg_i\end{smallmatrix}\right]}\right)$$
$$\left(\prod_{[g]\in(K\backslash H/K)_{sym}}\prod_{g_i\in K_{\pm g}\backslash H}\chi_g(v_1u_{g_i}(w))^{\left[\begin{smallmatrix}g_i\\gg_i\end{smallmatrix}\right]}\right).$$

By restricting $w\in W_E$, we get the first entry of $r_{\mathrm{g}}(w)$, namely

$$\begin{aligned} r_{\mathrm{g}}(w)_1 = &\left(\prod_{g\in(K\backslash H/K)_{asym/\pm}}\left(\prod_{g_i\in K_g\backslash K}\chi_g(u_{g_i}(w))\right)\left(\prod_{\substack{g_i\in K_g\backslash H\\ gg_i\in K}}\chi_g(u_{g_i}(w))^{-1}\right)\right)\\ &\left(\prod_{g\in(K\backslash H/K)_{sym}}\left(\prod_{g_i\in K_{\pm g}\backslash K}\chi_g(v_1u_{g_i}(w))\right)\left(\prod_{\substack{g_i\in K_{\pm g}\backslash H\\ gg_i\in K}}\chi_g(v_1u_{g_i}(w))^{-1}\right)\right). \end{aligned} \tag{12}$$

We now analysis the products in (12) and match them to those in (11). First, for $g \in (K\backslash H/K)_{asym/\pm}$, the first product of (12)

$$\prod_{g_i \in K_g\backslash K} u_{g_i}(w),\ w \in K$$

is the transfer map $T^K_{K_g} : K^{\mathrm{ab}} \to (K_g)^{\mathrm{ab}}$. By class field theory (see [Tat79]), it corresponds to the inclusion $E^\times \hookrightarrow E_g^\times$. Therefore

$$\prod_{g_i \in K_g\backslash K} \chi_g(u_{g_i}(w)) = \mathrm{Res}^{E_g^\times}_{E^\times} \chi_g(w),$$

which is the first factor in (11). Next we consider (the inverse of) the second product of (12)

$$\prod_{\substack{g_i \in K_g\backslash H \\ gg_i \in K}} u_{g_i}(w),\ w \in K$$

For $g_i \in K_g\backslash H$ such that $gg_i \in K$, we can write $g_i = g^{-1}x_i$ for some x_i running through a set in K of representatives of $K \cap gKg^{-1}\backslash K$. If u_{x_i} is the map (3) for $K \cap gKg^{-1}\backslash K$ then we have

$$g^{-1}(x_i w) = g^{-1}(u_{x_i}(w)x_{j(x_i,w)}), \tag{13}$$

where $u_{x_i(w)} \in K \cap gKg^{-1}$. On the other hand, by regarding $g^{-1}x_i \in K_g\backslash H$ we have

$$g^{-1}x_i w = u_{g^{-1}x_i}(w) g_{j(g^{-1}x_i,w)} \tag{14}$$

where $u_{g^{-1}x_i}(w) \in K_g$ and $g_{j(g^{-1}x_i,w)}$ is of the form $g^{-1}x_j$ for some j. By comparing (13) and (14) we have $g^{-1}u_{x_i}(w)g = u_{g^{-1}x_i}(w)$. Therefore

$$\prod_i u_{g_i}(w) = g^{-1}\left(\prod_i u_{x_i}(w)\right) g = g^{-1}T^K_{K\cap gKg^{-1}}(w)g$$

and hence

$$\prod_{\substack{g_i \in K_g\backslash H \\ gg_i \in K}} \chi_g(u_{g_i}(w)) = \chi_g^{g^{-1}}(T^K_{K\cap gKg^{-1}}(w)) = (\mathrm{Res}^{{}^g E_g^\times}_{E^\times} \chi_g^{g^{-1}})(w)$$

which is (the inverse of) the second factor in (11). Finally, for $g \in (K\backslash H/K)_{sym}$, we choose coset representatives $g_1, \ldots, g_k$, $gg_1, \ldots, gg_k$ for $K_g\backslash H$ such that $g_1, \ldots, g_k$ are those of $K_{\pm g}\backslash H$. Moreover we can assume that

$$g_1, \ldots, g_h, gg_{h+1}, \ldots, gg_{2h} \in K.$$

Hence the third product in (12) is

$$\prod_{\substack{g_i \in K_{\pm g}\backslash H \\ Kg_i = K}} \chi_g(v_1(u_{g_i}(w))) = \prod_{i=1}^{h} \chi_g(v_1(u_{g_i}(w))). \tag{15}$$

Here u_{g_i} is the map (3) for $K_{\pm g}\backslash H$ and so $v_1 u_{g_i}$ is the one for $K_g\backslash H$. For the fourth product in (12), because $\chi_g^g = \chi_g^{-1}$ (by condition (ii) of Definition 3.2) and $g(v_1(u_{g_i}(w)))g^{-1} = v_1(u_{gg_i}(w))$, we have indeed

$$\prod_{g_i \in K_{\pm g}\backslash H, gg_i \in K} \chi_g(v_1 u_{g_i}(w))^{-1} = \prod_{i=h+1}^{2h} \chi_g(v_1(u_{gg_i}(w))). \tag{16}$$

Therefore the product of (15) and (16) is $\chi_g(T^K_{K_g}(w)) = (\mathrm{Res}^{E_g^\times}_{E^\times} \chi_g)(w)$ which is the last factor of (11). $\square$

The product $\mu = \mu_{\{\chi_g\}}$ in Proposition 4.2, as $\{\chi_g\}$ runs through all χ-data, does not produce arbitrary character of $E^\times$. Its restriction on $F^\times$ has a specific form by Proposition 4.7. First recall the following known results. Given a group H we write 1_H the trivial representation of H. If K is a subgroup of H of finite index, denote $T_K^H : H^{\mathrm{ab}} \to K^{\mathrm{ab}}$ the transfer morphism. For any $g \in H$, we write ${}^gK = gKg^{-1}$.

Proposition 4.5. *Let σ and π be finite dimensional representations of K and H respectively. We have the following formulae.*

(i) (Mackey's Formula)

$$\mathrm{Res}_K^H \mathrm{Ind}_K^H \sigma \cong \bigoplus_{[g]\in K\backslash H/K} \mathrm{Ind}_{K\cap {}^gK}^K \mathrm{Res}_{K\cap {}^gK}^{{}^gK}({}^g\sigma).$$

(ii) $\det \mathrm{Ind}_K^H \sigma \cong (\det \mathrm{Ind}_K^H 1_K)^{\dim \sigma} \otimes (\det \sigma \circ T_K^H)$.

(iii) $(\det \mathrm{Res}_K^H \pi) \circ T_K^H = (\det \pi)^{|H/K|}$.

Proof. Formulae (i) and (ii) are well-known, for example (i) is proved in [Ser77] 7.3, and (ii) can be found in the Exercise in [Ser79] VII §8. Formula (iii) is direct from (ii) if we take $\sigma = \mathrm{Res}_K^H \pi$. □

In particular, if χ is a character of K, then by (ii) we have

$$\chi \circ T_K^H \cong \left(\det \mathrm{Ind}_K^H \chi\right) \left(\det \mathrm{Ind}_K^H 1_K\right). \tag{17}$$

Lemma 4.6. *We have the formula*

$$\det \mathrm{Ind}_K^H 1_K = \prod_{[g]\in (K\backslash H/K)'} \det \mathrm{Ind}_{K_g}^H 1_{K_g}.$$

Notice that $\det \mathrm{Ind}_{K_g}^H 1_{K_g}$ is independent of the choice of representative g of the double coset $[g]$ if we interpret the character as the sign of the canonical H-action on H/K_g, For all representatives of $[g]$, the corresponding actions are equivalent each other.

Proof. (of Lemma 4.6) Applying Mackey's formula on $\sigma = 1_K$ we obtain

$$\mathrm{Res}_K^H \mathrm{Ind}_K^H 1_K \cong \bigoplus_{[g]\in K\backslash H/K} \mathrm{Ind}_{K_g}^K 1_{K_g}.$$

We take determinant and then transfer morphism T_K^H on both sides. By (ii) and (iii) of Proposition 4.5 we obtain

$$\left(\det \mathrm{Ind}_K^H 1_K\right)^{|H/K|} = \prod_{[g]\in K\backslash H/K} \left(\det \mathrm{Ind}_{K_g}^H 1_{K_g}\right) \left(\det \mathrm{Ind}_K^H 1_K\right)^{|K/K_g|}.$$

Because the sum of $|K/K_g|$ for $[g]$ runs through $K\backslash H/K$ is $|H/K|$, the factor $\left(\det \mathrm{Ind}_K^H 1_K\right)$ on both sides vanish. What remains gives the desired formula. □

Proposition 4.7. *For all χ-data $\{\chi_g\}$, if μ is the character of $E^\times$ defined by $\{\chi_g\}$ as in Proposition 4.2, then $\mu|_{F^\times} \cong \det \mathrm{Ind}_{E/F} 1_{W_E}$.*

Proof. We first abbreviate $H = W_F$, $K = W_E$. For each $\lambda = \left[\begin{smallmatrix}1\\ g\end{smallmatrix}\right]$ we denote $K_g = K \cap {}^gK = W_{+\lambda}$ and $K_{\pm g} = W_{\pm\lambda}$. The isomorphism in Proposition 4.7 can be rewritten as

$$\prod_{[g]\in (K\backslash H/K)'} \chi_g \circ T_{K_g}^H = \det \mathrm{Ind}_K^H 1_K.$$

By Lemma 4.6 we have to show that

$$\prod_{[g]\in (K\backslash H/K)'} \chi_g \circ T_{K_g}^H = \prod_{[g]\in (K\backslash H/K)'} \det \mathrm{Ind}_{K_g}^H 1_{K_g} \tag{18}$$

By comparing (18) termwise, we claim that

(i) If $[g] \in (K\backslash H/K)_{asym/\pm}$, then

$$\left(\chi_g \circ T^H_{K_g}\right)\left(\chi_{g^{-1}} \circ T^H_{K_{g^{-1}}}\right) = \left(\det \mathrm{Ind}^H_{K_g} 1_{K_g}\right)\left(\det \mathrm{Ind}^H_{K_{g^{-1}}} 1_{K_{g^{-1}}}\right) \equiv 1.$$

(ii) If $[g] \in (K\backslash H/K)_{sym}$, then $\chi_g \circ T^H_{K_g} \equiv \det \mathrm{Ind}^H_{K_g} 1_{K_g}$.

If $[g] \in (K\backslash H/K)_{asym/\pm}$, then we have $K_{g^{-1}} = {}^gK_g$, which is the stabilizer of the root $\left[\begin{smallmatrix} 1 \\ g^{-1} \end{smallmatrix}\right]$. Because $\chi_{g^{-1}} = ({}^g\chi_g)^{-1}$ by condition (i) of Definition 3.2, we have

$$\left(\chi_g \circ T^H_{K_g}\right)\left(\chi_{g^{-1}} \circ T^H_{K_{g^{-1}}}\right) = \left(\chi_g \circ T^H_{K_g}\right)\left({}^g\chi_g^{-1} \circ T^H_{{}^gK_g}\right) \equiv 1.$$

On the other hand, since the H-action on ${}^H\left[\begin{smallmatrix} g \\ 1 \end{smallmatrix}\right] = {}^H\left[\begin{smallmatrix} 1 \\ g^{-1} \end{smallmatrix}\right]$ is equivalent to that on ${}^H\left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right]$, we have $\mathrm{Ind}^H_{K_g} 1_{K_g} \cong \mathrm{Ind}^H_{K_{g^{-1}}} 1_{K_{g^{-1}}}$. Therefore $\left(\det \mathrm{Ind}^H_{K_g} 1_{K_g}\right)\left(\det \mathrm{Ind}^H_{K_{g^{-1}}} 1_{K_{g^{-1}}}\right) \equiv 1$. We have proved the first claim. If $[g] \in (K\backslash H/K)_{sym}$, then we have an isomorphism $\mathrm{Ind}^{K_{\pm g}}_{K_g} 1_{K_g} \cong 1_{K_{\pm g}} \oplus \delta_{K_{\pm g}/K_g}$ as representations of $K_{\pm g}$. Here $\delta_{K_{\pm g}/K_g}$ is the quadratic character of $K_{\pm g}/K_g$. We denote this character by δ. Hence $\mathrm{Ind}^H_{K_g} 1_{K_g} \cong \mathrm{Ind}^H_{K_{\pm g}} 1_{K_{\pm g}} \oplus \mathrm{Ind}^H_{K_{\pm g}} \delta$ and

$$\det \mathrm{Ind}^H_{K_g} 1_{K_g} \cong (\det \mathrm{Ind}^H_{K_{\pm g}} 1_{K_{\pm g}})(\det \mathrm{Ind}^H_{K_{\pm g}} \delta) \tag{19}$$

by taking determinant. Now condition (ii) of Definition 3.2, namely $\chi_g \circ T^{K_{\pm g}}_{K_g} = \delta$, gives $\chi_g \circ T^H_{K_g} = \delta \circ T^H_{K_{\pm g}}$. By (17), this is just the right side of (19). We have proved the second claim and therefore Proposition 4.7. □

5 An application on Local Langlands Correspondence

We recall briefly the essentially tame local Langlands Correspondence, in the sense of [BH05, BH10]. Let F be a non-Archimedean local field of characteristic 0. Let G be GL_n as a reductive group over F. Let $\mathcal{A}^{et}_n(F)$ be the set of the isomorphism classes of irreducible essentially tame supercuspidal representations of $G(F)$, and $\mathcal{G}^{et}_n(F)$ be the set of the equivalence classes of essentially tame n-dimensional irreducible complex representations of W_F. The two notions of essential tameness above are defined in [BH05]. These two sets are bijective, whose map

$$\mathcal{L} = \mathcal{L}^{et}_n : \mathcal{G}^{et}_n(F) \to \mathcal{A}^{et}_n(F)$$

is called the essentially tame local Langlands Correspondence.

We introduce a collection for describing $\mathcal{L}$ explicitly. Let $P_n(F)$ be the set of W_F-equivalence classes (E, ξ) of admissible characters [BH05] ξ of $E^\times$ in which E/F is a tamely ramified extension of degree n. By [BH05] we know that $P_n(F)$ bijectively parameterizes both $\mathcal{A}^{et}_n(F)$ and $\mathcal{G}^{et}_n(F)$ simultaneously. Here the bijection $\sigma : P_n(F) \to \mathcal{G}^{et}_n(F)$ is simply induction of representations, while the one $\pi : P_n(F) \to \mathcal{A}^{et}_n(F)$ is constructed in [BK93] and [BH05]. The 'naive' Correspondence $\pi \circ \sigma^{-1} : \mathcal{G}^{et}_n(F) \to P_n(F) \to \mathcal{A}^{et}_n(F)$ does not satisfy certain conditions of the essentially tame local Langlands Correspondence (see Theorem 3.1 of [BH05] or Remark 5.2). In other words, the composition

$$\mu : P_n(F) \xrightarrow{\sigma} \mathcal{G}^{et}_n(F) \xrightarrow{\mathcal{L}} \mathcal{A}^{et}_n(F) \xrightarrow{\pi^{-1}} P_n(F)$$

does not give the identity map on $P_n(F)$. In [BH10] it is proved that for each admissible character ξ of $E^\times$, there is a character ${}_F\mu_\xi$ of $E^\times$, called the *rectifier* of ξ, such that ${}_F\mu_\xi \xi$ is also admissible and $\mu(E, \xi) = (E, {}_F\mu_\xi \xi)$.

In terms of admissible embeddings ${}^LT \to {}^LG$, it means that for each ξ we have to embed the image of the chosen Langlands parameter of ξ not by the canonical one defined by the Weyl group action (as in Remark 2.4) but the one twisted by its rectifier ${}_F\mu_\xi$. The rectifier ${}_F\mu_\xi$ is explicitly described in [BH10], and so is the Correspondence $\mathcal{L}$. Using this description we prove the following result in [Tam].

Theorem 5.1. *For each admissible ξ, the rectifier ${}_F\mu_\xi$ has a factorization of the form as in Theorem 1.2, for some canonical choice of χ-data.*

The proof requires a substantial amount of new concepts and computations, so it is better to deal with these in a separated article. This Theorem suggests that the rectifier has more properties inherited from that of χ-data. Indeed the symmetric structure of ${}_F\mu_\xi$ form that of χ-data $\{\chi_g\}$ is almost trivial because it is known [Tam] that those χ_g are of order at most 2 except for exactly one whose has order at most 4. We will have a closer look on this and also other inherited properties of ${}_F\mu_\xi$ in [Tam]. The following is a property which can be stated with the knowledge of this article.

Remark 5.2. Suppose $\sigma \in \mathcal{G}_n^{et}(F)$ and $\pi = \mathcal{L}(\sigma) \in \mathcal{A}_n^{et}(F)$. Let ω_π be the central character of π. One of the conditions of Langlands Correspondence, namely $\omega_\pi = \det \sigma$, implies that ${}_F\mu_\xi|_{F^\times} = \det \mathrm{Ind}_{E/F} 1_{W_E}$. This is a general fact about the restriction of the product of the characters in any χ-data as in Proposition 4.7, if we have established Theorem 5.1 beforehand. $\square$

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